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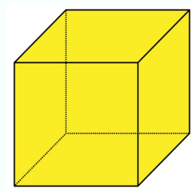
Recent Research at a glance

I work in a branch of mathematics known as *Algebraic Topology*. The objects of study are geometrical "shapes" which can exist in an arbitrary number of dimensions. They are known by various terms including: *spaces, manifolds, orbifolds* and *varieties*.

Though in general, these objects can't be visualized, mathematicians associate to them concrete algebraic structures known as *cohomology rings*. By manipulating the algebra, information is gleaned about similarities and differences among the shapes

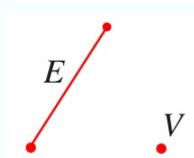
The algebra is understood best in cases where the recipe by which a space is made is controlled carefully. The Cartesian product is a relatively simple way of making complicated spaces from simpler ones. The case of a line interval I is illustrative. The Cartesian product $I \times I$ is square and the threefold Cartesian product $I \times I \times I$ is a cube.

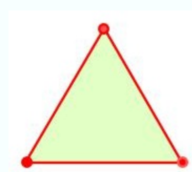
$$I \times I \times I \quad =$$



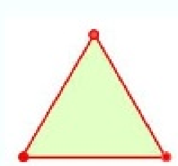
Here, by the simple construction of taking Cartesian products, a more geometrically rich space is made from a much simpler one.

Combinatorics allows for the introduction of further complexity into a Cartesian product. This is done by means of a combinatorial object known as a *simplicial complex*. Our purpose is to recognize certain spaces as *part* of a Cartesian product. In the example above, a threefold product, we consider simplicial complexes on three vertices, consisting of vertices, edges and triangles.

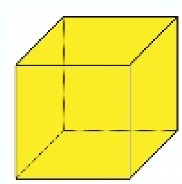




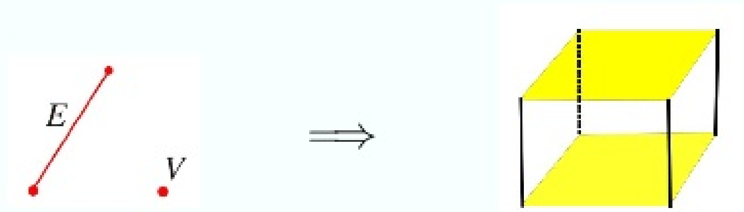
We think of the simplicial complex on the right, (the whole filled-in triangle), as encoding mathematically the *full* cube.



$$\Longrightarrow$$



The part of the triangle drawn below on the left, consisting of an edge E and an isolated vertex V, encodes a part of the cube which looks like the four-poster bed shown in the diagram below.



The precise manner in which a simplicial complex encodes part of a Cartesian product is somewhat involved. Roughly, the recipe is such that the edge E specifies the rectangles, top and bottom, and the single vertex Vspecifies the four "posts".

Notice that this object is part of the surface of the original cube, and from a topological point of view, its geometrical structure is more sophisticated than that of the original cube.

In a certain sense, the edge E and vertex V, comprise a combinatorial "polyhedron". For this reason, the part of the original Cartesian product determined by them, (the four-poster bed shape above), is called a *polyhedral product*.

The salient point here is that the algebra which gives information about the space which has been constructed, (the four-poster bed in this simple case), is computable in terms of the combinatorial information encoded in E and V. This becomes quite important when the space which is constructed is in large dimension.

A little more formally...

A polyhedral product $Z(K; (\underline{X}, \underline{A}))$ is a topological space determined by an abstract simplicial complex K on m vertices and a family of (based) CW pairs

$$(\underline{X}, \underline{A}) \;=\; \{(X_1, A_1), (X_2, A_2), \ldots, (X_m, A_m)\}.$$

It is defined as a certain colimit, (union), of Cartesian products inside $\prod_{i=1}^{m} X_i$, each parameterized by a simplex in K. In the case that K itself is a simplex,

$$Z(K; (X, A)) \;=\; \prod_{i=1}^{m} X_i.$$

The examples in the pictures above correspond to the case $(X_i, A_i) = (D^1, S^0)$.

Polyhedral products arose within the subject of toric topology, a topological approach to toric geometry, which had its genesis is a paper of Davis and Januszkiewicz. The basic spaces which arise were reformulated by Buchstaber and Panov into *moment–angle complexes* which correspond now to the case $(X_i, A_i) = (D^2, S^1)$.

In recent years, an extensive literature has developed as the framework of polyhedral products has found broad application throughout mathematics. Without being recognized as such, polyhedral products do have a history within homotopy theory but the current upsurge in activity is a direct consequence of the invention of moment–angle complexes.

This table lists a selection of current applications:

$(\underline{X}, \underline{A})$	$Z(K; (\underline{X}, \underline{A}))$
(D^2, S^1)	toric geometry and topology
(D^1, S^0)	surfaces, number theory, represention theory
(S^1, S^1_+)	robotics, arachnid mechanisms
$(S^1, *)$	right–angle Artin groups
$(\mathbb{R}\mathrm{P}^\infty, *)$	right–angle Coxeter groups
$(\mathbb{C}, \mathbb{C}^*)$	complements of coordinate arrangements
$(\mathbb{R}^n, (\mathbb{R}^n)^*)$	complements of certain non-coordinate arrangements
$(\mathbb{C}\mathrm{P}^\infty, \mathbb{C}\mathrm{P}^{\mathrm{k}})$	monomial ideal rings
(EG, G)	free groups
$(BG, *)$	monodromy, combinatorics, representation theory
$(PX, \Omega X)$	homotopy theory
$(S^{2k+1}, *)$	graph products, quadratic algebras

With my colleagues, I study the homotopy theory of polyhedral products and their application in toric topology/geometry. Lately, we've been thinking about the cohomology of orbifolds, and in particular, toric orbifolds.